

# Unconventional minimal subtraction and Bogoliubov-Parasyuk-Hepp-Zimmermann: massive scalar theory and critical exponents

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## Abstract

*We introduce a simpler although unconventional minimal subtraction renormalization procedure in the case of a massive scalar  $\lambda\phi^4$  theory in Euclidean space using dimensional regularization. We show that this method is very similar to its counterpart in massless field theory. In particular, the choice of using the bare mass at higher perturbative order instead of employing its tree-level counterpart eliminates all tadpole insertions at that order. As an application, we compute diagrammatically the critical exponents  $\eta$  and  $\nu$  at least up to two loops. We perform an explicit comparison with the Bogoliubov-Parasyuk-Hepp-Zimmermann (BPHZ) method at the same loop order, show that the proposed method requires fewer diagrams and establish a connection between the two approaches.*

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## I. INTRODUCTION

Methods in field theory are ubiquitous in several areas of theoretical physics. The advent of renormalization ideas [1] and the renormalization-group arguments [2] set the ground to various schemes in which one can extract finite (from otherwise meaningless infinite) results using perturbation theory. An important application is the computation of critical exponents from diagrammatic methods in a  $\lambda\phi^4$  theory which describes the universality classes of ordinary systems undergoing a phase transition [3]. During that early stage, the regularization method invented to handle properly the infinities due to ultraviolet divergences appearing in the computation of Feynman graphs utilized a momentum cutoff [4]. In addition, the utilization of dimensional regularization [5–7] together with minimal subtraction of dimensional poles led the subject of perturbative computation of exponents using massive fields to unprecedented precision within the  $\epsilon$ -expansion [8].

Minimal subtraction is much simpler when formulated in terms of massless fields. Consider only the multiplicatively renormalizable one-particle irreducible (1PI) vertex functions. The massless integrals are easier to evaluate since all polynomial dependence of a given diagram in the mass vanishes. Consequently, the minimal subtraction approach can be formulated without the need of the iterative construction of counterterms [9]. In case of a massive theory, one has to employ the technique named “partial  $p$ ” [5] in order to separate the dependence of polynomials in the mass from the contribution of polynomials in the external momenta (beside the contributions of logarithmical integrals combining both). A standard procedure is to employ the *BPHZ* renormalization method, which is the statement of renormalizability on the level of the Lagrangian [10–13]. The bare Lagrangian density includes the counterterms to be constructed perturbatively with appropriate vertices such that the theory is automatically renormalized. The counterterms determine three normalization constants, corresponding to the renormalization of the field, the mass and coupling constant, respectively. Within this scheme we have to compute a large number of diagrams. We can then ask ourselves whether we can find out a much simpler minimal subtraction version with a reduced number of diagrams in a certain higher order in the number of loops and verify its consistency with the rigorous but lengthier *BPHZ* technique through a simple application.

In this work we propose a new method of minimal subtraction of dimensional poles in a massive  $\lambda\phi^4$ . We restrict our discussion of the various vertex parts involved up to two-loop

level, except by the two-point function, which is going to be determined up to three-loop level. In the bare propagators, the tree-level bare mass is replaced by the three-loop bare mass. At two-loop level, the vertex parts which are required to renormalize the theory multiplicatively at this perturbative order work pretty much in the same manner as in the massless theory with the elimination of all tadpole graphs present in the diagrammatic expansion at the same perturbative order. We just need the renormalization functions of the field, the composite field and the expansion of the tree-level dimensionless bare coupling constant in terms of the renormalized dimensionless coupling. The nontrivial feature of this unconventional approach is that the two-point vertex part at three-loop order requires an extra renormalization, since the choice of the new bare mass parameter produces a residual single pole in that vertex function.

As an application, we compute the anomalous dimension of the field  $\eta$  at three-loop order as well as the correlation length exponent  $\nu$  at two-loop order by diagrammatic means. We then perform a detailed comparison with the traditional *BPHZ* method of minimal subtraction: we evaluated explicitly all the required diagrams, the fixed point and the corresponding Wilson functions at the fixed point. In spite of quite different intermediate results, we show that our method is much simpler since we only need to calculate a reduced number of diagrams. Comparing those results we are led to a dictionary between the two minimal subtraction renormalization schemes for this massive scalar field theory.

Section II presents all primitive divergent vertex parts required for the multiplicative renormalization along with their diagrammatic loop expansion. The minimal set of integrals (and their solutions as poles in  $\epsilon$ ) displayed in this section is outlined in Appendix A. Section III deals with the explicit renormalization of the vertex parts and how the extra subtraction in the two-point function at three-loops can be performed without changing the various normalization constants. We compute the Wilson functions, the fixed point and give a brief description of the computation of the critical exponents.

The *BPHZ* method is reviewed in Section IV. We calculate the critical exponents using this technique. Owing to simplicity, in Appendix B we evaluate one three-loop integral of the two-point function along with its counterterm in order to prove that the singular part of that combination of diagrams does not depend on the external momentum.

In Section V we discuss our proposal and include possible future applications in the concluding remarks.

## II. UNCONVENTIONAL MINIMAL SUBTRACTION FOR THE MASSIVE THEORY

Originally, the method which requires a minimal number of diagrams with the elimination of all tadpoles along the way using dimensional regularization was already discussed in Amit and Martin-Mayor's book [9] in the framework of renormalized massless fields. Our goal here is to adapt that technique to the massive renormalized theory. We shall employ that notation throughout this work.

The bare Lagrangian with  $O(N)$ -symmetry we are going to consider is given by

$$\mathcal{L} = \frac{1}{2} |\nabla \phi_0|^2 + \frac{1}{2} \mu_0^2 \phi_0^2 + \frac{1}{4!} \lambda (\phi_0^2)^2, \quad (1)$$

where  $\phi_0$ ,  $\mu_0$  and  $\lambda$  are the bare order parameter, mass ( $\mu_0^2 = t_0$  is the bare reduced temperature in statistical mechanics, which is proportional to  $\frac{T-T_C}{T_C}$ ) and coupling constant, respectively. Note that this expression tells us that we are going to proceed with our discussion in Euclidean space due to the signature chosen for the quadratic term in the derivatives. Our discussion will be founded entirely in momentum space. The primitively (bare) divergent one-particle irreducible (1PI) vertex parts which are required to renormalize the theory multiplicatively are the two-point vertex part  $\Gamma^{(2)}(k_1, k_2, \mu_0, \lambda)$ , the four-point vertex  $\Gamma^{(4)}(k_1, k_2, k_3, k_4, \mu_0, \lambda)$  and the two-point vertex point with composite field insertion  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda)$ , where  $k_i$  are the external momenta associated to the particular vertex part and  $Q_3$  is the external momentum of the insertion of the  $\phi^2$  composite operator. The reader should be aware that not all external momenta are independent.

Let us turn now to the multiplicative renormalizability statement in this formulation. One can define finite renormalized vertex parts out of the (infinite) bare vertex parts with arbitrary insertions of composite operators  $\Gamma^{(N,M)}(k_i; Q_j)$  where  $i = 1, \dots, N, j = 1, \dots, M$  ( $(N, M) \neq (0, 2)$ ) using renormalization functions. When the bare vertices are multiplied by the normalization functions of the field  $Z_\phi$  and of the composite field  $Z_{\phi^2}$  (whose divergences manifest themselves as inverse powers of  $\epsilon = 4 - d$ ), they yield the renormalized vertex parts

$$\Gamma_R^{(N,M)}(k_i; Q_j, g, m) = (Z_\phi)^{\frac{N}{2}} (Z_{\phi^2})^M \Gamma^{(N,M)}(k_i; Q_j), \quad (2)$$

which turn out to be finite. Note that in the left hand side of the last equation the parameters  $m$  and  $g$  are the renormalized mass and coupling constant, respectively.

Now let us discuss what are the bare parameters which enter in the right hand side of last equation. In order to do that, let us consider the loop expansion of the three vertex parts above mentioned.

Begin with  $\Gamma^{(2)}$ . Instead of writing the complete expression, let us draw the diagrams which correspond to all integrals that are going to be required in our subsequent discussion. The simplest diagrams which do not depend on the external momenta are the tadpoles and their cousins, which are given by the following expressions

$$\text{tadpole} = \frac{N+2}{3} \int \frac{d^d q}{q^2 + \mu_0^2}, \quad (3)$$

$$\text{tadpole with two loops} = \left(\frac{N+2}{3}\right)^2 \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)^2 (q_2^2 + \mu_0^2)}, \quad (4)$$

$$\text{tadpole with three loops} = \left(\frac{N+2}{3}\right)^3 \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu_0^2)^2 (q_2^2 + \mu_0^2)^2 (q_3^2 + \mu_0^2)}, \quad (5)$$

$$\text{tadpole with two loops and one external line} = \left(\frac{N+2}{3}\right)^3 \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu_0^2)^3 (q_2^2 + \mu_0^2)^2 (q_3^2 + \mu_0^2)}, \quad (6)$$

$$\text{tadpole with two loops and one external line (different)} = \left(\frac{N+2}{3}\right)^2 \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu_0^2)^2 (q_2^2 + \mu_0^2) (q_3^2 + \mu_0^2) ((q_1 + q_2 + q_3)^2 + \mu_0^2)}. \quad (7)$$

The other diagrams do depend on the external momenta and can be expressed in terms of integrals in the form

$$\text{tadpole with one external line} = \left(\frac{N+2}{3}\right) \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)^2 (q_2^2 + \mu_0^2) ((q_1 + q_2 + k)^2 + \mu_0^2)}, \quad (8)$$

$$\begin{aligned} \text{tadpole with two external lines} &= \left(\frac{(N+2)(N+8)}{27}\right) \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu_0^2) (q_2^2 + \mu_0^2) (q_3^2 + \mu_0^2)} \\ &\quad \times \frac{1}{((q_1 + q_2 + k)^2 + \mu_0^2) (q_1 + q_3 + k)^2 + \mu_0^2)}, \end{aligned} \quad (9)$$

$$\text{tadpole with two external lines (different)} = \left(\frac{N+2}{3}\right)^2 \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + \mu_0^2)^2 (q_2^2 + \mu_0^2) ((q_1 + q_2 + k)^2 + \mu_0^2) (q_3^2 + \mu_0^2)}. \quad (10)$$

Actually we shall not need all diagrams above displayed. In fact, our search is to choose a minimal set of graphs to work with. In a complete analogy with the massless framework, we could think of considering only two of these diagrams, namely Eqs. (8) and (9).

Denoting the bare propagator  $(k^2 + \mu_0^2)^{-1}$  by a line, we can write a symbolic expression of the diagrammatic expansion involving  $\Gamma^{(2)}$  at three-loop order with respect to our previous

graphs. We obtain:

$$\begin{aligned} \Gamma^{(2)}(k, \mu_0, \lambda) = & \text{---}^{-1} + \frac{\lambda}{2} \text{---} \bigcirc \text{---} - \frac{\lambda^2}{4} \text{---} \bigcirc \bigcirc \text{---} - \frac{\lambda^2}{6} \text{---} \bigcirc \text{---} + \frac{\lambda^3}{4} \text{---} \bigcirc \bigcirc \bigcirc \text{---} \\ & + \frac{\lambda^3}{4} \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \frac{\lambda^3}{12} \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \frac{\lambda^3}{8} \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \frac{\lambda^3}{8} \text{---} \bigcirc \bigcirc \bigcirc \text{---}. \end{aligned} \quad (11)$$

Next, define the three-loop bare mass parameter through the expression  $\mu = \Gamma^{(2)}(k = 0, \mu_0, \lambda)$ , or explicitly

$$\begin{aligned} \mu^2 = \mu_0^2 & + \frac{\lambda}{2} \text{---} \bigcirc \text{---} - \frac{\lambda^2}{4} \text{---} \bigcirc \bigcirc \text{---} - \frac{\lambda^2}{6} \text{---} \bigcirc \text{---} \Big|_{k=0} + \frac{\lambda^3}{4} \text{---} \bigcirc \bigcirc \bigcirc \text{---} \Big|_{k=0} \\ & + \frac{\lambda^3}{4} \text{---} \bigcirc \bigcirc \bigcirc \text{---} \Big|_{k=0} + \frac{\lambda^3}{12} \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \frac{\lambda^3}{8} \text{---} \bigcirc \bigcirc \bigcirc \text{---} + \frac{\lambda^3}{8} \text{---} \bigcirc \bigcirc \bigcirc \text{---}. \end{aligned} \quad (12)$$

Performing the inversion to express  $\mu_0$  in terms of  $\mu$ , we get to an expansion for  $\Gamma^{(2)}(k, \mu_0, \lambda)$ , which depends implicitly on  $\mu$ . The tadpoles graphs and their cousins will vanish into the  $\Gamma^{(2)}(k, \mu_0, \lambda)$  expression obtained after the substitution  $\mu_0(\mu)$ . What remain are the diagrams Eqs. (8), (9), (10) subtracted from their counterparts computed at  $k = 0$ . Expanding  $\mu_0$  up to first order in the coupling constant inside the “sunset” diagrams, it follows that

$$\text{---} \bigcirc \text{---} \Big|_{\mu_0} = \text{---} \bigcirc \text{---} \Big|_{\mu} + \frac{3\lambda}{2} \text{---} \bigcirc \bigcirc \bigcirc \text{---} \Big|_{\mu}. \quad (13)$$

By replacing this expression into  $\Gamma^{(2)}(k, \mu_0, \lambda)$ , the diagram corresponding to Eq. (10) is eliminated. Second, we find out an expression which no longer depends on  $\mu_0$ , namely

$$\begin{aligned} \Gamma^{(2)}(k, \mu_0, \lambda) = k^2 + \mu^2 & - \frac{\lambda^2}{6} \left( \text{---} \bigcirc \text{---} \Big|_{\mu} - \text{---} \bigcirc \text{---} \Big|_{k=0, \mu} \right) \\ & + \frac{\lambda^3}{4} \left( \text{---} \bigcirc \bigcirc \bigcirc \text{---} \Big|_{\mu} - \text{---} \bigcirc \bigcirc \bigcirc \text{---} \Big|_{k=0, \mu} \right). \end{aligned} \quad (14)$$

The two-point vertex function now depends explicitly on the three-loop bare mass  $\mu$ . We can write this transmutation as  $\Gamma(k, \mu_0, \lambda) \equiv \Gamma(k, \mu, \lambda)$  and no longer have to make any reference to the tree-level bare mass. What occurs to the other primitively divergent vertex parts when we perform a similar manipulation on their graphs?

Consider the four-point vertex part. Its diagrams up to two-loop order are given by

$$\left[ \text{Diagram 1} \right] (k) = \frac{(N+8)}{9} \int \frac{d^d q}{(q^2 + \mu_0^2)((q+k)^2 + \mu_0^2)}, \quad (15)$$

$$\begin{aligned} \left[ \text{Diagram 2} \right] (k) &= \frac{N^2 + 6N + 20}{27} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((q_1+k)^2 + \mu_0^2)(q_2^2 + \mu_0^2)} \\ &\quad \times \frac{1}{((q_2+k)^2 + \mu_0^2)}, \end{aligned} \quad (16)$$

$$\begin{aligned} \left[ \text{Diagram 3} \right] (k_i) &= \left( \frac{5N+22}{27} \right) \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((k_1+k_2-q_1)^2 + \mu_0^2)} \\ &\quad \times \frac{1}{(q_2^2 + \mu_0^2)((k_3+q_1-q_2)^2 + \mu_0^2)}, \end{aligned} \quad (17)$$

$$\left[ \text{Diagram 4} \right] (k) = \frac{(N+2)(N+8)}{27} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((q_1+k)^2 + \mu_0^2)(q_2^2 + \mu_0^2)}. \quad (18)$$

Henceforth, we denote a permutation of external momenta on vertex parts which depend upon them by “*perm.*”. The expansion of the four-point vertex function can be written pictorially as

$$\begin{aligned} \Gamma^{(4)}(k_i, \mu_0, \lambda) &= \lambda - \frac{\lambda^2}{2} \left( \left[ \text{Diagram 1} \right] (k_1+k_2) + 2\text{perms.} \right) \\ &+ \frac{\lambda^3}{4} \left( \left[ \text{Diagram 2} \right] (k_1+k_2) + 2\text{perms.} \right) + \frac{\lambda^3}{2} \left( \left[ \text{Diagram 3} \right] (k_i) + 5\text{perms.} \right) \\ &+ \frac{\lambda^3}{2} \left( \left[ \text{Diagram 4} \right] (k_1+k_2) + 2\text{perms.} \right). \end{aligned} \quad (19)$$

With the substitution of the initial bare mass in the propagators of all these graphs as a function of the new bare mass  $\mu$ , the first diagram will produce itself calculated with  $\mu$  plus a correction which exactly cancels the last term. After the replacement  $\mu_0 \rightarrow \mu$  we find

$$\begin{aligned} \Gamma^{(4)}(k_i, \mu_0, \lambda) &= \lambda - \frac{\lambda^2}{2} \left( \left[ \text{Diagram 1} \right]_{\mu} (k_1+k_2) + 2\text{perms.} \right) \\ &+ \frac{\lambda^3}{4} \left( \left[ \text{Diagram 2} \right]_{\mu} (k_1+k_2) + 2\text{perms.} \right) \\ &+ \frac{\lambda^3}{2} \left( \left[ \text{Diagram 3} \right]_{\mu} (k_i) + 5\text{perms.} \right). \end{aligned} \quad (20)$$

The four-point vertex part “does not remember” the original dependence on  $\mu_0$  after the elimination of extra diagrams and we are left with the minimal set of its diagrams with  $\mu_0 \rightarrow \mu$ . If we interpret  $\Gamma^{(4)}(k_i, \mu, \lambda) (\equiv \Gamma^{(4)}(k_i, \mu, \lambda))$  at two loops as a two-loop truncation of the value of  $\mu$ , it is consistent even though the bare mass is defined at three-loop order.

Finally, the diagrams of the bare vertex  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda)$  up to two-loop order are given by:

$$\left[ \text{Diagram 1} \right] (k) = \frac{N+2}{18} \int \frac{d^d q}{(q^2 + \mu_0^2)((q+k)^2 + \mu_0^2)}, \quad (21)$$

$$\left[ \text{Diagram 2} \right] (k) = \frac{(N+2)^2}{54} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((q_1+k)^2 + \mu_0^2)(q_2^2 + \mu_0^2)}, \quad (22)$$

$$\left[ \text{Diagram 3} \right] (k) = \frac{(N+2)^2}{108} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((q_1+k)^2 + \mu_0^2)} \times \frac{1}{(q_2^2 + \mu_0^2)((q_2+k)^2 + \mu_0^2)}, \quad (23)$$

$$\left[ \text{Diagram 4} \right] (k_1, k_2; Q_3) = \frac{(N+2)}{36} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + \mu_0^2)((k_1+k_2-q_1)^2 + \mu_0^2)} \times \frac{1}{(q_2^2 + \mu_0^2)((Q_3+q_1-q_2)^2 + \mu_0^2)}. \quad (24)$$

The diagrammatic expansion for the vertex  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda)$  can be written as

$$\begin{aligned} \Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda) &= 1 - \lambda \left( \left[ \text{Diagram 1} \right] (k_1 + k_2) + 2\text{perms.} \right) \\ &+ \lambda^2 \left( \left[ \text{Diagram 2} \right] (k_1 + k_2) + 2\text{perms.} \right) + \lambda^2 \left( \left[ \text{Diagram 3} \right] (k_1 + k_2) + 2\text{perms.} \right) \\ &+ \lambda^2 \left( \left[ \text{Diagram 4} \right] (k_1, k_2; Q_3) + 5\text{perms.} \right). \end{aligned} \quad (25)$$

In those graphs all the propagators are evaluated with  $\mu_0$  up to now. Now expanding  $\mu_0^2(\mu)$  as before, the  $O(\lambda)$  first nontrivial diagrams originate themselves with  $\mu_0$  replaced by  $\mu$  along with  $O(\lambda^2)$  corrections which eliminate precisely the second diagrams. We then get to the following expression:

$$\begin{aligned} \Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda) &= 1 - \lambda \left( \left[ \text{Diagram 1} \right]_{\mu} (k_1 + k_2) + 2\text{perms.} \right) \\ &+ \lambda^2 \left( \left[ \text{Diagram 3} \right]_{\mu} (k_1 + k_2) + 2\text{perms.} \right) + \lambda^2 \left( \left[ \text{Diagram 4} \right]_{\mu} (k_1, k_2; Q_3) + 5\text{perms.} \right) \end{aligned} \quad (26)$$

We conclude that the same desirable feature goes on again: the bare vertex  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda)$  is insensitive to  $\mu_0$  and if we define  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu, \lambda) \equiv \Gamma^{(2,1)}(k_1, k_2; Q_3, \mu_0, \lambda)$ , this vertex is also reduced to the minimal number of diagrams with propagator involving only the new bare mass parameter  $\mu$ .



Any potentially divergent vertex part which can be renormalized multiplicatively includes the three primitively divergent vertex parts just discussed (skeleton expansion [14]). Thus, the renormalization of the three vertex parts which include only the minimal number of diagrams will be sufficient to minimally renormalize the vertex function under consideration. Now we can make explicit reference to the argument of the bare vertex functions. At first, there would be new multiplicative renormalizability should be a statement that by considering a given bare theory *with bare mass*  $\mu$  at a certain perturbative order (and a given tree-level bare coupling constant), the renormalized vertex functions should be finite and satisfy

$$\Gamma_R^{(N,M)}(k_i; Q_j, g, m) = (Z_\phi)^{\frac{N}{2}} (Z_{\phi^2})^M \Gamma^{(N,M)}(k_i; Q_j, \mu, \lambda). \quad (27)$$

From now on we are going to determine the normalization functions  $Z_\phi$  and  $Z_{\phi^2}$  using the results of the computation of the Feynman diagrams in Appendix A. Due to our explicit treatment of the bare vertex part  $\Gamma^{(2,1)}(k_1, k_2; Q_3, \mu, \lambda)$ , we define the quantity  $\bar{Z}_{\phi^2} = Z_{\phi^2} Z_\phi$  which shall be useful in our manipulations.

We follow a trend which is standard in the computation of critical exponents using this language [9]. Nevertheless, we shall see in a moment that this new technique does bring new insight in renormalization theory.

### III. RENORMALIZATION FUNCTIONS AND CRITICAL EXPONENTS IN UNCONVENTIONAL MINIMAL SUBTRACTION

Here we are going to calculate explicitly the critical exponents. The mainstream of our presentation will be brief, since this material is standard in literature [9, 14]. However, we shall unveil the role of the extra subtraction in the renormalized mass and its relationship with certain integrals which will come up in our discussion.

The renormalized theory possesses a flow in parameter space generated by the renormalized mass  $m$ : the same bare theory may give origin to many renormalized theories with different renormalized masses. The renormalization group flow of the coupling constant in parameter space is generated by the function  $\beta(g, m) = m \frac{\partial g}{\partial m}$ . In order to discard undesirable dimensionful parameters when  $d = 4 - \epsilon$ , define the Gell-Mann-Low function  $[\beta(g, \mu)]_{GL} = -\epsilon g + \beta(g, \mu)$ . Hence, even away from the critical dimension we are able to get rid of all dimensional couplings defining them in terms of dimensionless couplings as

$\lambda = \mu^\epsilon u_0$  and  $g = \mu^\epsilon u$ , where  $\mu$  is the bare mass at the loop order considered. By using the Gell-Mann-Low function inside the Callan-Symanzik equation, the description follows entirely in terms of dimensionless coupling constant. Those definitions imply that the object  $[\beta(g, \mu)]_{GL} \frac{\partial}{\partial g} = \beta(u) \frac{\partial}{\partial u}$  has a well defined scaling limit [15, 16].

After collecting these steps together we are left with the perturbative computation of the Wilson functions

$$\beta(u) = -\epsilon \left( \frac{\partial \ln u_0}{\partial u} \right), \quad (28a)$$

$$\gamma_\phi(u) = \beta(u) \left( \frac{\partial \ln Z_\phi}{\partial u} \right), \quad (28b)$$

$$\gamma_{\phi^2}(u) = -\beta(u) \left( \frac{\partial \ln Z_{\phi^2}}{\partial u} \right), \quad (28c)$$

$$\bar{\gamma}_{\phi^2}(u) = -\beta(u) \left( \frac{\partial \ln \bar{Z}_{\phi^2}}{\partial u} \right) = -\gamma_\phi(u) + \gamma_{\phi^2}(u). \quad (28d)$$

We first write the primitively divergent bare vertex expansion in terms of the minimal set of Feynman diagrams previously defined in the form

$$\Gamma^{(2)}(k, u_0, \mu) = k^2 + \mu^2 - B_2 \mu^{2\epsilon} u_0^2 + B_3 \mu^{3\epsilon} u_0^3, \quad (29a)$$

$$\Gamma^{(4)}(k_i, u_0, \mu) = \mu^\epsilon u_0 [1 - A_1 \mu^\epsilon u_0 + (A_2^{(1)} + A_2^{(2)}) \mu^{2\epsilon} u_0^2], \quad (29b)$$

$$\Gamma^{(2,1)}(k_1, k_2; p, u_0, \mu) = 1 - C_1 \mu^\epsilon u_0 + (C_2^{(1)} + C_2^{(2)}) \mu^{2\epsilon} u_0^2. \quad (29c)$$

Since the main modification with respect to standard minimal subtraction schemes in the computation of critical exponents is related to the two-point vertex part, we consider it as our starting point. It is obvious from Eqs. (14) along with Eqs. (A28), (A31) and (A32) from Appendix A that

$$B_2 = \frac{N+2}{18} \mu^{-2\epsilon} \left\{ -\frac{P^2}{8\epsilon} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] - \frac{3\mu^2}{4} \tilde{I}(P) \right\}. \quad (30)$$

Furthermore, comparing Eq. (29a) with the diagrammatic expansion (14), we obtain the following symbolic result

$$B_3 = \frac{1}{4} \left( \left( \text{diagram 1} \right) \Big|_\mu - \left( \text{diagram 2} \right) \Big|_{k=0, \mu} \right). \quad (31)$$

Now using Eq. (9) in conjunction with Eq. (A42), we find

$$B_3 = \left( \frac{(N+2)(N+8)}{108} \right) \mu^{-3\epsilon} \left\{ -\frac{P^2}{6\epsilon^2} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] - \frac{5\mu^2}{2\epsilon} \tilde{I}(P) \right\}. \quad (32)$$

Before proceeding, let us define the dimensionless bare couplings and the renormalization functions in minimal subtraction as powers series in the renormalized dimensionless coupling constant in the form

$$u_0 = u[1 + \sum_{i=1}^{\infty} a_i(\epsilon)u^i], \quad (33a)$$

$$Z_\phi = 1 + \sum_{i=1}^{\infty} b_i(\epsilon)u^i, \quad (33b)$$

$$\bar{Z}_{\phi^2} = 1 + \sum_{i=1}^{\infty} c_i(\epsilon)u^i. \quad (33c)$$

By requiring minimal subtraction of dimensional poles, the renormalized primitively divergent vertex parts should be finite order by order in powers of  $u$ . This in turn determines  $a_i(\epsilon)$ ,  $b_i(\epsilon)$  and  $c_i(\epsilon)$ . For the sake of simplicity, consider the renormalized vertices

$$\Gamma_R^{(2)}(k, u, m) = Z_\phi \Gamma^{(2)}(k, u_0, \mu), \quad (34a)$$

$$\Gamma_R^{(4)}(k_i, u, m) = Z_\phi^2 \Gamma^{(4)}(k_i, u_0, \mu), \quad (34b)$$

$$\Gamma_R^{(2,1)}(k_1, k_2, p; u, m) = \bar{Z}_{\phi^2} \Gamma^{(2,1)}(k_1, k_2, p, u_0, \mu), \quad (34c)$$

up to two-loop order (neglect the  $B_3$  coefficient in the bare vertex  $\Gamma^{(2)}(k, u, \mu)$ ). First, replace the expansion for  $Z_\phi$  into  $\Gamma_R^{(2)}(k, u, m)$  and define  $m^2 \equiv Z_\phi \mu^2$  in the mass term which does not multiply coupling constant factors. Next, the value  $u_0 = u$  can surely be taken at this order. Recalling that regular terms are not taken into account in this set of steps, we find

$$\Gamma_R^{(2)}(k, u, m) = k^2 + m^2 + k^2 b_1 u + k^2 u^2 \left( b_2 + \frac{(N+2)}{144\epsilon} \right). \quad (35)$$

The absence of poles in  $\epsilon$  requires that  $b_1 = 0$ , which is consistent with the absence of the tadpole graph and  $b_2 = -\frac{(N+2)}{144\epsilon}$ .

Focusing our attention now in  $\Gamma^{(4)}(k, u_0, \mu)$ , the coefficients appearing in its bare counterpart can be written in terms of the integrals computed in Appendix A, namely

$$A_1 = \frac{(N+8)}{18} [I_2(k_1 + k_2) + I_2(k_1 + k_3) + I_2(k_2 + k_3)], \quad (36a)$$

$$A_2^{(1)} = \frac{(N^2 + 6N + 20)}{108} [I_2^2(k_1 + k_2) + I_2^2(k_1 + k_3) + I_2^2(k_2 + k_3)], \quad (36b)$$

$$A_2^{(2)} = \frac{(5N + 22)}{54} [I_4(k_i) + 5perms.]. \quad (36c)$$

Since  $Z_\phi^2 = 1 + 2b_2u^2 + O(u^4)$ ,  $\Gamma_R^{(4)}(k_i, u, m)$  can be expressed in the form

$$\begin{aligned} \Gamma_R^{(4)}(k_i, u, m) = & \mu^\epsilon(1 + 2b_2u^2)(u + a_1u^2 + a_2u^3)[1 - A_1\mu^\epsilon(u + a_1u^2) \\ & + (A_2^{(1)} + A_2^{(2)})\mu^{2\epsilon}u^2], \end{aligned} \quad (37)$$

which can be simplified by grouping the powers of  $u$  together. This leads to

$$\begin{aligned} \Gamma_R^{(4)}(k_i, u, m) = & \mu^\epsilon(u + (a_1 - A_1\mu^\epsilon)u^2 + (a_2 + 2b_2 - 2a_1A_1\mu^\epsilon \\ & + (A_2^{(1)} + A_2^{(2)})\mu^{2\epsilon})u^3). \end{aligned} \quad (38)$$

By demanding that the poles be minimally cancelled at  $O(u^3)$ , we employ Eqs. (A4), (A5), (A7) and (A13) combined with Eqs. (36). In the resulting expression, all integrals  $\tilde{L}(P)$  which appear in the several loop contributions of the four-point vertex function vanish and we obtain the following singular coefficients:

$$a_1 = \frac{(N+8)}{6\epsilon}, \quad (39a)$$

$$a_2 = \frac{(N+8)^2}{36\epsilon^2} - \frac{(3N+14)}{24\epsilon}. \quad (39b)$$

Let us complete our task at two-loop level by analyzing the vertex part  $\Gamma^{(2,1)}$  in the computation of  $\bar{Z}_{\phi^2}$ . When we utilize Eqs. (21)-(24) together with Eq. (26), we can identify the coefficients present in Eq. (29c) as

$$C_1 = \frac{N+2}{18}[I_2(k_1+k_2) + I_2(k_1+k_3) + I_2(k_2+k_3)], \quad (40a)$$

$$C_2^{(1)} = \frac{(N+2)^2}{108}[I_2^2(k_1+k_2) + I_2^2(k_1+k_3) + I_2^2(k_2+k_3)], \quad (40b)$$

$$C_2^{(2)} = \frac{N+2}{36}[I_4(k_i) + 5perms.]. \quad (40c)$$

Employing Eq. (34c) in conjunction with the expansions of  $\bar{Z}_{\phi^2}$  and  $u_0$  in powers of  $u$ , we are led to

$$\begin{aligned} \Gamma^{(2,1)}(k_1, k_2; p, u_0, \mu) = & 1 + (c_1 - C_1\mu^\epsilon)u + (c_2 - (c_1 + a_1)\mu^\epsilon C_1 \\ & + (C_2^{(1)} + C_2^{(2)})\mu^{2\epsilon})u^2. \end{aligned} \quad (41)$$

Requirement of minimal cancellations of the poles allows to compute  $c_1$  and  $c_2$ . Indeed, the integral  $\tilde{L}(P)$  attached to  $I_2$  and  $I_4$  (see Appendix A) cancels out similarly to what took

place in the renormalization of the  $\Gamma^{(4)}$  vertex part, resulting in the coefficients:

$$c_1 = \frac{(N+2)}{6\epsilon}, \quad (42a)$$

$$c_2 = \frac{(N+2)(N+5)}{36\epsilon^2} - \frac{(N+2)}{24\epsilon}. \quad (42b)$$

So far everything is entirely similar to what happens in the minimal subtraction scheme for massless fields.

Let us turn now our attention to the computation of  $Z_\phi$  at three-loop level, i.e., we have to compute  $b_3$  by using the diagrammatic expansion considering up to the  $B_3$  contribution in  $\Gamma^{(2)}$ . Performing all the steps just like before, we have to be careful to use  $u_0^2 = u^2 + \frac{(N+8)}{3\epsilon}u^3$ . This produces a nontrivial mixing involving the  $B_2$  and  $B_3$  diagrams, which in the end of the day eliminates the  $\tilde{L}_3(k)$  integrals appearing in both terms. Notice that  $b_3$  is determined from the coefficient of the  $k^2$  term. Working out the details, we find

$$b_3 = -\frac{(N+2)(N+8)}{1296\epsilon^2} + \frac{(N+2)(N+8)}{5184\epsilon}. \quad (43)$$

However, this is not sufficient to subtract all the poles in  $\epsilon$ , since the resulting vertex part at this order is given by

$$\begin{aligned} \Gamma_R^{(2)}(k, u, m) = & k^2 + m^2 \left[ 1 + \frac{(N+2)}{24} u^2 \tilde{I}(k) \right. \\ & \left. - \frac{(N+2)(N+8)}{\epsilon} u^3 \left( \frac{\tilde{I}(k)}{108} \right) \right], \end{aligned} \quad (44)$$

where from the Appendix A we know that

$$\tilde{I}(k) = \int_0^1 dx \int_0^1 dy \ln y \frac{d}{dy} \left( (1-y) \ln \left[ \frac{y(1-y)\frac{k^2}{\mu^2} + 1 - y + \frac{y}{x(1-x)}}{1 - y + \frac{y}{x(1-x)}} \right] \right). \quad (45)$$

Although the integral is well behaved, we seem to be in trouble here, since the advertised minimal subtraction has produced a residual pole at three-loop level in contradiction with the minimal subtraction assertion in the first place! We already know that the theory is renormalizable by minimal subtraction had we used the tree-level bare mass  $\mu_0$  and a larger number of diagrams. What is happening is obvious: the price to pay for a reparametrization  $\mu_0 \rightarrow \mu$  which eliminates many diagrams is the appearance of the above integral which seems to invalidate the minimal subtraction procedure only at three-loop order.

The way out to this difficulty is the introduction of an extra subtraction in the two-point vertex function, in order to compensate for the bare mass reparametrization, which removes

the remaining pole and we are done. Thus, by defining

$$\tilde{\Gamma}_R^{(2)}(k, u, m) = \Gamma_R^{(2)}(k, u, m) + m^2 \left[ \frac{(N+2)(N+8)}{\epsilon} u^3 \left[ \frac{\tilde{I}(k)}{108} \right] \right], \quad (46)$$

we establish a direct connection with the minimal subtraction in the massless theory, since those terms proportional to  $m^2$  are not there and this maneuver is unnecessary. In comparison with normalization conditions in the massive theory, at  $k = 0$   $\tilde{\Gamma}_R^{(2)}(0, u, m) = \Gamma_R^{(2)}(0, u, m) = m^2$  and we do not need to perform the extra subtraction.

Thence, our new proposal requires the vertex parts  $\tilde{\Gamma}_R^{(2)}(k, u, m)$ ,  $\Gamma_R^{(4)}(k_i, u, m)$  and  $\Gamma_R^{(2,1)}(k_1, k_2, p; u, m)$  to be renormalized by minimal subtraction with the same normalization functions  $Z_\phi$  and  $\bar{Z}_{\phi^2}$  from the original renormalized field theory.

We are now in position to derive the exponents from our evaluation of the Wilson functions. In terms of the coefficients from  $Z_\phi$ ,  $\bar{Z}_{\phi^2}$  and  $u_0$ , they are given by

$$\beta = -\epsilon u [1 - a_1 u + 2(a_1^2 - a_2)u^2], \quad (47a)$$

$$\gamma_\phi = -\epsilon u [2b_2 u + (3b_3 - 2b_2 a_1)u^2], \quad (47b)$$

$$\bar{\gamma}_{\phi^2} = \epsilon u [c_1 + (2c_2 - c_1^2 - a_1 c_1)u]. \quad (47c)$$

The eigenvalue condition  $\beta(u_\infty) = 0$  defines the repulsive fixed point  $u_\infty$ . It results in the following expression

$$u_\infty = \frac{6}{8+N} \epsilon \left\{ 1 + \epsilon \frac{(9N+42)}{(8+N)^2} \right\}. \quad (48)$$

Replacing this fixed point value we obtain the anomalous dimension of the field which is identical to the exponent  $\eta \equiv \gamma_\phi(u_\infty)$ , namely

$$\eta = \frac{1}{2} \epsilon^2 \frac{N+2}{(N+8)^2} \left[ 1 + \epsilon \left( \frac{6(3N+14)}{(N+8)^2} - \frac{1}{4} \right) \right], \quad (49)$$

whereas the quantities  $\gamma_{\phi^2}(u_\infty)$  and  $\bar{\gamma}_{\phi^2}(u_\infty)$  are given by the expressions

$$\gamma_{\phi^2}(u_\infty) = \frac{N+2}{(N+8)} \epsilon \left[ 1 + \frac{(13N+44)}{2(N+8)^2} \epsilon \right], \quad (50a)$$

$$\bar{\gamma}_{\phi^2}(u_\infty) = \frac{N+2}{(N+8)} \epsilon \left[ 1 + \frac{6(N+3)}{(N+8)^2} \epsilon \right]. \quad (50b)$$

They are related to the anomalous dimension of the composite operator, also known as the correlation length exponent  $\nu$ , through the expression  $\nu^{-1} = -d_{\phi^2} = 2 - \bar{\gamma}_{\phi^2}(u_\infty) - \gamma_\phi(u_\infty) (= 2 - \gamma_{\phi^2}(u_\infty))$ . Consequently, we can read off its value as

$$\nu = \frac{1}{2} + \frac{(N+2)}{4(N+8)} \epsilon + \frac{1}{8} \frac{(N+2)(N^2+23N+60)}{(N+8)^3} \epsilon^2. \quad (51)$$

These critical exponents correspond to those previously found using: *i*) massless fields within either the minimal subtraction or the normalization conditions formulation and *ii*) massive theory using normalization conditions.

The aforementioned results show that in spite of the extra subtraction that has to be carried over the vertex  $\Gamma_R^{(2)}$  in our new procedure, the exponents are really the same when we use a minimal set of Feynman graphs in the determination of the renormalization functions. This minimal subtraction can be viewed as the counterpart of the normalization conditions framework *ii*) from [6]. The relationship between the two massive formulations is entirely analogous to that shared by massless fields in *i*).

In order to compare with another standard method of minimal subtraction, we shall turn our attention to the BPHZ method in the next section which works with a much larger set of diagrams.

#### IV. BPHZ IN MINIMAL SUBTRACTION

The BPH method involves the iterative introduction of counterterms in the original bare Lagrangian density in order to renormalize all vertex functions. These counterterms are required to obey the symmetries of the original bare Lagrangian. A very good account of this subject can be found in the book by Kleinert and Schulte-Frohlinde [17].

We shall follow that material hereafter but point out some different conventions adopted in the present work. First, the symbol  $\mathcal{K}()$  to be applied in a diagram  $()$  in order to pick out the singular part is going to be replaced by  $()_S$ . We only utilize this notation for one-loop diagrams and tadpoles insertions on them. In the two- and three-loop diagrams of the two-point function, we include some regular parts depending on logarithmic integrals in order to show how they cancel out in the renormalization algorithm. The other higher-loop diagrams will be displayed only with their singular parts. In this sense our operationalization of the method in the present section resembles the approach described in (chapter 9 of) that book within the spirit of Refs. [10, 12]. Second, since we are going to restrict ourselves to a fixed loop order where we already know all the diagrams involved, we shall not make use of the formal  $R$ -operation which permits the generation of counterterms at arbitrary order in perturbation theory, albeit this procedure can also be applied and shown to be equivalent to the  $BPH$  construction [13]. Thus, we found appropriate to call this technique  $BPHZ$

method. Third, the multiplicative factor appearing in each loop integral in the conventions of Kleinert and Schulte-Frohlinde's book is different from the conventions adopted here (see [9] for a similar notation as ours).

We shall maintain the notation presented so far, performing some adaptations from the method discussed in that book. In order to do so, we start with a quick description of the script necessary to carry out the computation of the critical exponents.

Using the bare Lagrangian density Eq. (1) and performing the redefinitions  $\phi_0 = Z_\phi^{\frac{1}{2}}\phi$ ,  $\mu_0^2 = m^2 \frac{Z_{m^2}}{Z_\phi}$  and  $\lambda = \frac{Z_u}{Z_\phi^2} \mu^\epsilon u$ , it turns out to be given by

$$\mathcal{L} = \frac{1}{2} Z_\phi |\nabla \phi|^2 + \frac{1}{2} m^2 Z_{m^2} \phi^2 + \frac{1}{4!} \mu^\epsilon u Z_u (\phi^2)^2, \quad (52)$$

whose coefficients are  $\epsilon$ -dependent. Note that the mass scale  $\mu$  is arbitrary,  $m$  is the renormalized mass,  $u$  is the renormalized dimensionless coupling constant and  $Z_\phi = 1 + \delta_\phi$ ,  $Z_{m^2} = 1 + \delta_{m^2}$  and  $Z_u = 1 + \delta_u$  are the renormalization functions. The amounts  $\delta_\phi$ ,  $\delta_{m^2}$  and  $\delta_u$  are the counterterms which are added at each diagram in arbitrary loop order in order to cancel the singular contributions of the primitively divergent bare vertex parts. Denote the external momentum by  $P$ . The counterterms generate additional vertices and originate the following Feynman rules in momentum space

$$\text{---}\bigcirc\text{---} = P^2 \delta_\phi, \quad (53a)$$

$$\text{---}\bigtimes\text{---} = m^2 \delta_{m^2}, \quad (53b)$$

$$\text{---}\bullet\text{---} = \mu^\epsilon u \delta_u. \quad (53c)$$

In practice we shall need these counterterms expanded in powers of  $u$  up to the desired order. In the present work, the counterterms will be expanded in the following form:  $\delta_\phi = \delta_\phi^{(1)} u + \delta_\phi^{(2)} u^2 + \delta_\phi^{(3)} u^3$ ,  $\delta_{m^2} = \delta_{m^2}^{(1)} u + \delta_{m^2}^{(2)} u^2$  and  $\delta_u = \delta_u^{(1)} u + \delta_u^{(2)} u^2$ .

In the present section we shall not need to determine the higher loop two-point function graphs in the degree of detail presented in Appendix A. As it is going to be shown in the remainder, only a simplified form of that vertex part suitable to our purposes will be worked out explicitly. Every graph in the present method is computed with the renormalized mass, but all integrals appearing depend on dimensionless ratios such as  $\frac{k^2}{\mu^2}$  or  $\frac{m^2}{\mu^2}$  which always show up in intermediate steps in the argument of logarithms. At least in a particular loop order, we shall verify that all of them cancel in the end of a sample calculation.



We start with the tadpole diagram. The corresponding integral is

$$I_T = \int d^d q \frac{1}{q^2 + m^2}. \quad (54)$$

Using Eq. (A2), absorbing  $S_d$  in the redefinition of the coupling constant, performing the continuation  $d = 4 - \epsilon$  and the expansion up to  $O(\epsilon^0)$  we find

$$I_T = -\frac{m^2}{\epsilon} \left[ 1 - \frac{\epsilon}{2} \ln(m^2) \right]. \quad (55)$$

Therefore, when we include the  $O(N)$  factor, the tadpole diagram is given by the following expression

$$\text{tadpole} = -\frac{(N+2)}{3} \frac{m^2}{\epsilon} \left[ 1 - \frac{\epsilon}{2} \ln(m^2) \right]. \quad (56)$$

The double tadpole is characterized by the integral

$$I_{DT} = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + m^2)^2 (q_2^2 + m^2)}. \quad (57)$$

The integrals over  $q_1$  and  $q_2$  can be performed independently. The former is identical to a four-point one-loop diagram at zero external momenta, whereas the latter is given by Eq. (56) discussed above. Using these facts, one can show that the double tadpole graph is represented by the expression

$$\text{double tadpole} = -\left(\frac{N+2}{3}\right)^2 \frac{m^2}{\epsilon^2} \left[ 1 - \frac{\epsilon}{2} - \epsilon \ln(m^2) \right]. \quad (58)$$

The integral corresponding to the sunset diagram was already computed in Appendix A. In the notation of the present section it is given by

$$I_3(P, m) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + m^2)(q_2^2 + m^2)[(q_1 + q_2 + P)^2 + m^2]}.$$

When we employ a simpler form of its calculation sketched in Appendix A multiplied to the  $O(N)$  factor, the sunset symbol is equivalent to the expression

$$\text{sunset} = -\left(\frac{N+2}{3}\right) \left( \frac{3m^2}{2\epsilon^2} \left[ 1 + \frac{1}{2}\epsilon - \epsilon \ln(m^2) \right] + \frac{P^2}{8\epsilon} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon L_3(P) \right] \right), \quad (59)$$

where

$$L_3(P) = \int_0^1 dx dy (1-y) \ln \left\{ y(1-y)P^2 + \left[ 1 - y + \frac{y}{x(1-x)} \right] m^2 \right\}. \quad (60)$$

The three-loop contribution to the two-point function

$$I_5(P, m) = \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{(q_1^2 + m^2)(q_2^2 + m^2)(q_3^2 + m^2)} \times \frac{1}{[(q_1 + q_2 + P)^2 + m^2][(q_1 + q_3 + P)^2 + m^2]},$$

was also computed in the Appendix A. Its solution in a form useful to our purposes in the present section conjugated to the symmetry factor associated to the  $O(N)$  symmetry implies that the associated graph is given by

$$\begin{aligned} \text{Diagram} &= -\frac{(N+2)(N+8)}{27} \left( \frac{5m^2}{3\epsilon^3} \left[ 1 + \epsilon \left( 1 - \frac{3}{2} \ln(m^2) \right) + \epsilon^2 \left( \frac{\pi^2}{24} + \frac{15}{4} + \frac{9}{8} (\ln(m))^2 \right) \right. \right. \\ &\quad \left. \left. + \frac{3}{2} \tilde{i}(P) \right) \right] + \frac{P^2}{6\epsilon^2} \left[ 1 + \frac{1}{2} \epsilon - 3\epsilon L_3(P) \right] \right). \end{aligned} \quad (61)$$

Since the last expression is  $O(u^3)$ , the coefficient of the  $\epsilon^{-3}$  part which is proportional to the mass only contribute to the mass counterterm at  $O(u^3)$  which is of an order higher than we need in our present discussion. The combination of this term with those coming from three-loop tadpole diagrams is certainly important in the proof of mass renormalization at three-loops. We shall neglect it consistently with the arguments to be explained next.

The remaining three-loop diagrams from the two-point vertex can be separated in two distinct sets. The first one corresponds to tadpole and four-point insertions into tadpole diagrams (Eqs. (5)-(7)) and do not depend upon the external momenta. Together with their counterterms, their singular part will contribute to the mass renormalization at three-loop level. Remember that we are interested only in the computation of  $Z_\phi$  (proportional to  $P^2$ ) up to three-loop order. We do not have to consider those diagrams for they will not contribute to the evaluation of  $Z_\phi$ . The second set corresponds solely to the “sunset” diagram with a tadpole insertion Eq. (10) and its counterterm. In the Appendix B, we show explicitly that the singular parts (poles in  $\epsilon$ ) coming from these integrals do not depend on the external momenta and also do not contribute to the computation of  $Z_\phi$ . We are going to consider them implicitly in the three-loop expansion of the two-point vertex part but shall not work them out from now on. They are going to be collectively referred to as “tadpoles” in the remainder of this section.

Next we shall analyze the graphs contributing to the four-point function. The one-loop integral


$$I_2(P, m) = \int d^d q \frac{1}{(q^2 + m^2)[(q + P)^2 + m^2]},$$

can be read off from Eq. (A4) from Appendix A, which in conjunction with Eq. (15) produces the following result to its corresponding diagram:

$$\text{Diagram} = \frac{(N+8)}{9} \frac{1}{\epsilon} \left[ 1 - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon L(P) \right], \quad (62)$$

where

$$L(P) = \int_0^1 dx \ln[x(1-x)P^2 + m^2]. \quad (63)$$

The integral associated to the diagram  and given by

$$\int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + m^2)[(q_1 + P)^2 + m^2]} \frac{1}{(q_2^2 + m^2)[(q_2 + P)^2 + m^2]},$$

can be written diagrammatically as the square of the previous one-loop contribution. Therefore, we obtain the following expression


$$\text{Diagram} = \frac{(N^2 + 6N + 20)}{27} \frac{1}{\epsilon^2} [1 - \epsilon - \epsilon L(P)]. \quad (64)$$

Consider the nontrivial two-loop contribution

$$I_4(P, m) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + m^2)[(P - q_1)^2 + m^2](q_2^2 + m^2)[(q_1 - q_2 + P)^2 + m^2]}.$$

In order to compare with Eq. (A13), we just have to replace the bare mass by the renormalized one  $m$  without factoring it out from the integral. It then follows that the solution to its diagram can be written as

$$\text{Diagram} = \left( \frac{5N + 22}{27} \right) \frac{1}{2\epsilon^2} \left[ 1 - \frac{1}{2}\epsilon - \epsilon L(P) \right]. \quad (65)$$

Another two-loop diagram contributing to the four-point vertex part is , which is represented by the integral

$$I_{2T}(P, m) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + m^2)^2 [(q_1 + P)^2 + m^2] (q_2^2 + m^2)}.$$

The integral over  $q_2$  is simply a tadpole, whereas the integral over  $q_1$  can be evaluated using Feynman parameters. Only the singular part of its diagram will be interesting to our purposes and turns out to be

$$\left[ \text{Diagram} \right]_S = - \frac{(N+2)(N+8)}{27} \left[ \frac{m^2}{2\epsilon} \int_0^1 dx \frac{(1-x)}{x(1-x)P^2 + m^2} \right]. \quad (66)$$

The method from last section considered the diagrammatic expansion without counterterms and the renormalization functions were obtained by demanding finite renormalized vertex functions. Here, the normalization functions are obtained directly from the counterterms generated order by order in perturbation theory. We start by using the diagrammatic expansion of the two-point vertex function up to two-loop order, which including counterterms diagrams, reads

$$\begin{aligned} \Gamma^{(2)}(P, m, \mu^\epsilon u) = & P^2 + m^2 + u \left( \frac{\mu^\epsilon}{2} \text{---}\bigcirc\text{---} + m^2 \delta_{m^2}^{(1)} + P^2 \delta_\phi^{(1)} \right) + u^2 \left( - \frac{\mu^{2\epsilon}}{4} \text{---}\bigcirc\text{---}\bigcirc\text{---} \right. \\ & - \frac{\mu^{2\epsilon}}{6} \text{---}\bigcirc\text{---} - \frac{\mu^\epsilon m^2 \tilde{\lambda}_{m^2}}{2u} \text{---}\bigcirc\text{---} + \frac{\mu^\epsilon \tilde{\lambda}_u}{2u} \text{---}\bigcirc\text{---} + m^2 \delta_{m^2}^{(2)} + P^2 \delta_\phi^{(2)} \Big), \end{aligned} \quad (67)$$

where the last two diagrams are computed at zero external momentum since they are constructed out from tadpoles and shall be discussed in a moment. As the counterterms select just the singular part ( $\equiv ()_S$ ) of the diagrams, the conditions of finiteness of this vertex part at one-loop order ( $O(u)$ ) are equivalent to the following identifications

$$m^2 \delta_{m^2}^{(1)} = -\frac{\mu^\epsilon}{2} \left( \text{---}\bigcirc\text{---} \right)_S, \quad (68a)$$

$$P^2 \delta_\phi^{(1)} = -P^2 \frac{\mu^\epsilon}{2} \left( \text{---}\bigcirc\text{---} \right)_S, \quad (68b)$$

which in conjunction with Eq. (56), lead to the following coefficients

$$\delta_{m^2}^{(1)} = \frac{(N+2)}{6\epsilon}, \quad (69a)$$

$$\delta_\phi^{(1)} = 0. \quad (69b)$$

Now consider the fourth graph in Eq. (67). It is a double tadpole where the upper tadpole was replaced by its counterterm. In other words, the upper counterterm coupling constant can be identified through the relation  $\tilde{\lambda}_{m^2} \equiv u \delta_{m^2}^{(1)}$  and the diagram can be expressed in the form:

$$\frac{\mu^\epsilon m^2 \tilde{\lambda}_{m^2}}{2u} \text{---}\bigcirc\text{---} = m^2 \frac{(N+2)^2}{36\epsilon^2} \left[ 1 - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon \ln \left( \frac{m^2}{\mu^2} \right) \right]. \quad (70)$$

Before going ahead, consider the one-loop 4-point vertex function. Its diagrammatic expansion including the counterterm is given by

$$\Gamma^{(4)}(k_i, m, \mu^\epsilon u) = u \mu^\epsilon \left( 1 - u \frac{\mu^\epsilon}{2} \left( \text{---}\bigcirc\text{---} \right) (k_1 + k_2) + 2 \text{ permutations} \right) + u \delta_u^{(1)}. \quad (71)$$

By using Eq. (62), this vertex part is finite if the following relation holds:

$$\delta_u^{(1)} = \frac{(N+8)}{6\epsilon}. \quad (72)$$

The fifth (last counterterm) diagram of the two-point function in Eq. (67) is a product of a tadpole with a four-point insertion, where the latter loop has shrunk to zero but picking out the coupling constant from its singular counterterm. This is equivalent to take  $\tilde{\lambda}_u \equiv u\delta_u^{(1)}$  and its corresponding expression reads:

$$\frac{\mu^\epsilon \tilde{\lambda}_u}{2u} \text{---}\bigcirc \text{---} = -m^2 \frac{(N+2)(N+8)}{36\epsilon^2} \left[ 1 - \frac{\epsilon}{2} \ln \left( \frac{m^2}{\mu^2} \right) \right]. \quad (73)$$

Now we replace the results of this discussion in Eq. (67) in order to determine  $\delta_{m^2}^{(2)}$  and  $\delta_\phi^{(2)}$  at two-loop level. Indeed, substitution of the Eqs. (58), (59), (70) and (73) into Eq. (67) followed by the expansion in powers of  $\mu^{n\epsilon}$  up to first order in  $\epsilon$ , we find that all the terms  $\ln \left( \frac{m^2}{\mu^2} \right)$  cancel out at  $O(u^2)$ . We then obtain:

$$\delta_{m^2}^{(2)} = \frac{(N+2)(N+5)}{36\epsilon^2} - \frac{(N+2)}{24\epsilon}, \quad (74a)$$

$$\delta_\phi^{(2)} = -\frac{(N+2)}{144\epsilon}. \quad (74b)$$

Let us examine the two-loop contribution from  $\Gamma^{(4)}$ . We just have to focus on the two-loop diagrams in order to determine the counterterm  $\delta_u^{(2)}$ , namely

$$\begin{aligned} \Gamma_{2-loop}^{(4)}(k_i, m, \mu^\epsilon u) &= \mu^\epsilon u^3 \left[ \frac{\mu^{2\epsilon}}{4} \left( \left[ \text{---}\bigcirc \text{---}\bigcirc \text{---} \right] (k_1 + k_2) + 2perms. \right) \right. \\ &+ \frac{\mu^{2\epsilon}}{2} \left( \left[ \text{---}\bigcirc \text{---} \right] (k_i) + 5perms. \right) + \frac{\mu^{2\epsilon}}{2} \left( \left[ \text{---}\bigcirc \text{---} \right] (k_1 + k_2) + 2perms. \right) \\ &+ \frac{\mu^\epsilon m^2 \tilde{\lambda}_{m^2}}{2u} \left( \left[ \text{---}\bigcirc \text{---} \right] (k_1 + k_2) + 2perms. \right) - \frac{\mu^\epsilon \tilde{\lambda}_u}{u} \left( \left[ \text{---}\bigcirc \text{---} \right] (k_1 + k_2) + 2perms. \right) \\ &\left. + \delta_u^{(2)} \right]. \end{aligned} \quad (75)$$

The counterterm diagram corresponding to the fourth type of graphs appearing in the last expansion is given by

$$\begin{aligned} \left[ \text{---}\bigcirc \text{---} \right] (k) &= \frac{(N+8)}{9} \int d^d q_1 \frac{1}{(q^2 + m^2)^2 [(q+k)^2 + m^2]} \\ &= \left[ \frac{(N+8)}{18} \int_0^1 dx \frac{(1-x)}{x(1-x)k^2 + m^2} \right] + O(\epsilon). \end{aligned} \quad (76)$$

The multiplication of these diagrams by the factor  $\frac{\mu^\epsilon m^2 \tilde{\lambda}_{m^2}}{2u}$  have a singular part which cancels exactly the contribution of the third kind of diagrams. What is left is the requirement that

$\delta_u^{(2)}$  should subtract minimally the poles of the two-loop diagrams from  $\Gamma_{2-loop}^{(4)}(k_i, m, \mu^\epsilon u)$ , i.e.,

$$\delta_u^{(2)} = \frac{\mu^{2\epsilon}}{4} \left( \left[ \text{diagram 1} \right] (k_1 + k_2) + 2 \text{ perms.} \right)_S + \frac{\mu^{2\epsilon}}{2} \left( \left[ \text{diagram 2} \right] (k_i) + 5 \text{ perms.} \right)_S - \frac{\mu^\epsilon \tilde{\lambda}_u}{u} \left( \left[ \text{diagram 3} \right] (k_1 + k_2) + 2 \text{ perms.} \right)_S. \quad (77)$$

The last diagram is just a one-loop diagram of the four-point coupling constant with its associated symmetry factor attached. When it is multiplied by  $\frac{\mu^\epsilon \tilde{\lambda}_u}{u}$ , with  $\tilde{\lambda}_u = u\delta_u^{(1)}$ , we find

$$\frac{\mu^\epsilon \tilde{\lambda}_u}{u} \left( \left[ \text{diagram 3} \right] (k_1 + k_2) \right)_S = \frac{(N+8)^2}{54\epsilon^2} \left( 1 - \frac{\epsilon}{2} - \frac{\epsilon}{2} \hat{L}(k_1 + k_2) \right), \quad (78)$$

where

$$\hat{L}(P) = \int_0^1 dx \ln \left[ \frac{x(1-x)P^2 + m^2}{\mu^2} \right]. \quad (79)$$

When we expand the factors  $\mu^{n\epsilon}$  in powers of logarithms, the integral  $L(P)$  in Eq. (63) gets transformed to  $\hat{L}(P)$  in all diagrams appearing in Eq. (77). Summing up everything utilizing Eqs. (64) and (65) in conjunction with the last expressions, we verify that all terms proportional to  $\hat{L}(P)$  with  $(P = k_1 + k_2, k_1 + k_3 \text{ and } k_2 + k_3)$  vanish. Therefore, this manipulation produces the following result

$$\delta_u^{(2)} = \frac{(N+8)^2}{36\epsilon^2} - \frac{(5N+22)}{36\epsilon}. \quad (80)$$

The three-loop diagrams of the two-point vertex part include “tadpoles” as well as relevant graphs to our computation of  $\delta_\phi^{(3)}$ . We can get rid of the former setting  $m = 0$  into their solution, which at the same time eliminates contributions to the mass renormalization at three-loops (proportional to  $m^2$ ) and we do not have to worry about those terms at this perturbative order. For instance, the symbol  $(\text{diagram})_{m^2=0}$  implement the last condition.

The three-loop diagrams of  $\Gamma^{(2)}$ , for the purpose of computing  $\delta_\phi^{(3)}$ , can be written in a simplified form as

$$\Gamma_{3-loop}^{(2)}(P) = u^3 \left( \frac{\mu^{3\epsilon}}{4} \left[ \text{diagram 4} \right]_{m^2=0} - \frac{\mu^{2\epsilon} \tilde{\lambda}_u}{3u} \left[ \text{diagram 5} \right]_{m^2=0} + P^2 \delta_\phi^{(3)} + \text{tadpoles} \right), \quad (81)$$

and from now on we are going to neglect the contributions coming from the tadpoles. Notice that even if in the remaining two diagrams the terms proportional to  $m^2$  are going to be

set to zero, we are not going to employ this simplification in the terms proportional to  $P^2$ , since we still have to demonstrate the elimination of  $L_3(P)$ -type contributions. Indeed, the object which appears when the  $\mu^{n\epsilon}$  coefficient is expanded is given by:

$$\hat{L}_3(P) = \int_0^1 dx dy (1-y) \ln \left\{ \frac{y(1-y)P^2 + \left[1 - y + \frac{y}{x(1-x)}\right]m^2}{\mu^2} \right\}. \quad (82)$$

Combining the last equation with the definition of  $\tilde{\lambda}_u$ , the solution of the diagrams represented by Eqs. (59), (61) and recalling the above remarks one can show that the  $\hat{L}_3(P)$  contributions vanish. What remains after the cancellation of the dimensional poles is the identification of the normalization coefficient:

$$\delta_\phi^{(3)} = -\frac{(N+2)(N+8)}{1296\epsilon^2} \left(1 - \frac{\epsilon}{4}\right). \quad (83)$$

Therefore, the complete solution at the loop order required for the three normalization functions is represented by

$$Z_\phi = 1 - \frac{(N+2)}{144\epsilon} u^2 - \frac{(N+2)(N+8)}{1296\epsilon^2} \left(1 - \frac{\epsilon}{4}\right) u^3, \quad (84a)$$

$$Z_{m^2} = 1 + \frac{(N+2)}{6\epsilon} u + \left[ \frac{(N+2)(N+5)}{36\epsilon^2} - \frac{(N+2)}{24\epsilon} \right] u^2, \quad (84b)$$

$$Z_u = 1 + \frac{(N+8)}{6\epsilon} u + \left[ \frac{(N+8)^2}{36\epsilon^2} - \frac{(5N+22)}{36\epsilon} \right] u^2. \quad (84c)$$

We have at hand the tools required to calculate the critical exponents. We start by the definition

$$\beta(u) = \mu \left( \frac{\partial u}{\partial \mu} \right)_{[\mu_0, \lambda]} = -\mu \left[ \frac{(\frac{\partial \lambda}{\partial \mu})_{(\mu_0, u)}}{(\frac{\partial \lambda}{\partial u})_{(\mu_0, \mu)}} \right]. \quad (85)$$

Remember that  $\lambda = Z_u Z_\phi^{-2} \mu^\epsilon u$ . Then, it follows directly that

$$\beta(u) = -\epsilon \frac{\partial \ln[Z_u Z_\phi^{-2} u]^{-1}}{\partial u}. \quad (86)$$

Utilizing Eqs. (84a)-(84b), we can rewrite last expression in terms of the coefficients just obtained after some algebra as

$$\beta = -\epsilon u [1 - \delta_u^{(1)} u + 2((\delta_u^{(1)})^2 - \delta_u^{(2)} + 2\delta_\phi^{(2)}) u^2]. \quad (87)$$

It is important to mention that this expression is formally different from Eq. (47a), since the individual terms in the latter after convenient identifications are not identical. Nevertheless,

in terms of the explicit computations already performed last equation can be written as

$$\beta(u) = -\epsilon u + \frac{(N+8)}{6}u^2 - \frac{(3N+14)}{12}u^3, \quad (88)$$

which is exactly the same result that would have been obtained from the explicit substitution of the coefficients into Eq. (47a).

Next, define the quantity  $\gamma_\phi(u) = \mu \left( \frac{\partial \ln Z_\phi}{\partial \mu} \right)_{[\mu_0, \lambda]} = \beta(u) \left( \frac{\partial \ln Z_\phi}{\partial u} \right)$ . In terms of the various coefficients, it is given by

$$\gamma_\phi(u) = -\epsilon u [2\delta_\phi^{(2)}u + (3\delta_\phi^{(3)} - 2\delta_u^{(1)}\delta_\phi^{(2)})u^2], \quad (89)$$

which is equivalent to

$$\gamma_\phi(u) = \frac{(N+2)}{72}u^2 - \frac{(N+2)(N+8)}{1728}u^3. \quad (90)$$

The last two equations are identical to those from our previous unconventional description. Now we introduce the amount  $\gamma_m(u) = \frac{\mu}{m^2} \left( \frac{\partial m^2}{\partial \mu} \right)_{[\mu_0, \lambda]} = \gamma_\phi(u) - \beta(u) \left( \frac{\partial \ln Z_{m^2}}{\partial u} \right)$ . It is convenient also to employ the notation  $\bar{\gamma}_m(u) = -\beta(u) \left( \frac{\partial \ln Z_{m^2}}{\partial u} \right)$ . It is easy to show that

$$\gamma_m(u) = \gamma_\phi(u) + \epsilon u [\delta_{m^2}^{(1)} + (2\delta_{m^2}^{(2)} - (\delta_{m^2}^{(1)})^2 - \delta_u^{(1)}\delta_{m^2}^{(1)})u]. \quad (91)$$

We can work this out further in order to obtain the simpler expression

$$\gamma_m(u) = \frac{(N+2)}{6}u \left[ 1 - \frac{5}{12}u \right]. \quad (92)$$

The nontrivial fixed point is given by the eigenvalue condition  $\beta(\tilde{u}_\infty) = 0$ , namely

$$\tilde{u}_\infty = \frac{6\epsilon}{(N+8)} \left[ 1 + \frac{3(3N+14)\epsilon}{(N+8)^2} \right]. \quad (93)$$

It turns out that  $\gamma_\phi(\tilde{u}_\infty)$  is simply the anomalous dimension of the field, exponent  $\eta$  from Eq. (49), namely

$$\eta = \gamma_\phi(\tilde{u}_\infty). \quad (94)$$

Moreover, at the fixed point we find out that

$$\gamma_m(\tilde{u}_\infty) = \frac{N+2}{(N+8)}\epsilon \left[ 1 + \frac{(13N+44)}{2(N+8)^2}\epsilon \right]. \quad (95)$$

Note that this expression is identical to Eq. (50a) for the Wilson function of the composite field. The above definitions get transliterated in the previous unconventional minimal subtraction through the identifications  $\gamma_m(u) = \gamma_{\phi^2}(u)$  and  $\bar{\gamma}_m(u) = \bar{\gamma}_{\phi^2}(u)$ . The exponent  $\nu$  is related to  $\gamma_m(\tilde{u}_\infty)$  through the relation  $\nu = (2 - \gamma_m(\tilde{u}_\infty))^{-1}$ . Consequently, it is simple to demonstrate that  $\nu$  obtained from this expression is identical to the result from Eq. (51).



## V. DISCUSSION OF THE RESULTS AND CONCLUSION

It is worthy to mention that the universal results coming from the new unconventional subtraction method introduced in Secs. II and III and the BPHZ method discussed in the Section IV are identical as expected, even though the intermediate steps are quite different.

Rigorously speaking, the BPHZ does not require any regularization method, since it is designed to yield finite expressions from any diagram by subtracting the divergent part without specification to the regulator employed. We used dimensional regularization in order to compare the same technique with previous results obtained using different conventions [17]. We restricted ourselves only to the (large number of) diagrams strictly necessary to perform the explicit calculations of all quantities required in the computation of the critical exponents at the desired order in perturbation theory.

The unconventional method, on the other hand, requires an extra subtraction for the two-point function beyond three-loop order due to our choice of using the three-loop bare mass instead of employing the tree-level bare mass. The advantage is that all the tadpoles diagrams do not need to be considered in this framework, since they drop out trivially after expanding the bare propagator in terms of the new bare mass inside all diagrams.

It is interesting to notice that despite the extra subtraction above mentioned, we can relate the method rather simply to minimal subtraction in the massless theory and to the massive theory using normalization conditions at the same order in perturbation theory. In fact, since the extra subtraction in the new method is proportional to the renormalized mass, at zero mass this subtraction is identically zero. Of course, in that case the mass scale  $\mu$  is replaced by a external nonvanishing momentum scale  $\kappa$  and the cancellations involving integrals of logarithms of the external momentum in the massless case carry out in the same way as in the unconventional method. Second, we recover the normalization conditions for the massive theory at zero external momentum with a minimal number of diagrams as proposed in [18], which is also rather similar to the massless case.

The choice of the three-loop bare mass to compute Feynman diagrams implies that only beyond three-loops we need the extra subtraction at the two-point vertex function, since everything works in exactly the same way at two-loop order whether we manipulate the minimal set of diagrams or if we utilize the full set of graphs. In particular, the extra subtraction involves an integral which is essentially different from the logarithmic integrals

which also can multiply poles in  $\epsilon$ . The last integrals do not vanish at zero external momenta, while the new integral presented is identically zero at vanishing external momenta. Although it is certainly not polynomial in the external momenta, it does behave as a polynomial in the external momentum and can be tacitly identified with a “harmless pole” [5] (which, rigorously speaking, is defined only when the residue of the pole is a polynomial of finite order in the external momenta) and the extra subtraction becomes natural within this context.

The consistency of the unconventional method is warranted when we confront it with its BPHZ standard counterpart. Along with the match of the field normalization constant in both formalisms, the identification of the composed field with the mass renormalization constant is exact. The beta functions are different in both frameworks but the combinations of the several diagrams produce the same answer: they yield the same fixed point. The other Wilson functions are proved to be the same in both schemes which lead to the same critical exponents.

For higher loops we expect that more differences between these two methods can appear. A generic feature should be the appearance of more extra subtractions at the two-point function vertex part consisting of integrals that behave themselves as harmless poles. As discussed, this complication is directly connected with the definition of the bare mass at that loop order. On the other hand, the simplification achieved in the elimination of all tadpole insertions in the vertex parts required in the perturbative calculation of the critical exponents at that order compensates this extra subtraction.

The present unconventional minimal subtraction procedure is completely different from traditional resummation methods designed to extract numerical estimations from the perturbative computation of physical quantities [17]. In our approach, the normalization constants are obtained in the weak-coupling limit as explained above. The analysis in the remainder amounts to take the physical critical system at the (repulsive) fixed point keeping, however, a nonvanishing renormalized mass which prevents the system of going to the strong-coupling regime. It is also distinct in comparison with variational perturbation theory [17, 19], for in that method the weak-coupling renormalization constants are obtained in the conventional way, the bare coupling constant is taken to infinity (renormalized mass tends to zero at the attractive massless fixed point) characterizing the strong-coupling regime where the resummation is then defined in  $d = 4 - \epsilon$ . The comparison with three-dimensional systems could

then be done by choosing either  $\epsilon = 1$  or by starting from scratch with fixed dimension  $d = 3$  and performing numerically the Feynman integrals and computing the critical exponents in the strong-coupling regime of small masses but without explicit scale invariance [20]. We emphasize that our unconventional resummation here just obtains the  $\epsilon$ -expansion results for the exponents. The aforementioned conventional resummations involving the  $\epsilon$ -expansion could be applied to our approach (involving the renormalized vertex parts  $\tilde{\Gamma}_R^{(2)}$ ,  $\Gamma_R^{(4)}$ ,  $\Gamma_R^{(2,1)}$ , and  $u_0(u)$ ) since they represent an extra ingredient in ameliorating the convergence properties of the  $\epsilon$ -expansion as far as numerical estimations (e.g., of critical exponents) are concerned.

Applications of the unconventional as well as the standard BPHZ minimal subtraction methods could be employed in some problems involving the formulation of critical phenomena using massive scalar field theories. For instance, in calculating critical exponents or other universal quantities of ordinary finite size systems in a parallel plate layered geometry [21, 22]. In addition, the investigation of those techniques could shed new light in massive  $\phi^4$  scalar theories when two mass scales are present as is the case in anisotropic  $m$ -axial Lifshitz criticalities [23]. This is a natural extension of the formalism discussed in the present paper. Since the massive theory is more appealing in its connection with quantum field theory, it would be interesting to investigate the perturbative analysis concerning field theories in Lifshitz spacetimes (see, for example, Refs. [24] and [25]). This constitutes a nice prelude to the treatment of a similar problem with several mass scales appearing naturally in the context of anisotropic generic competing systems of the Lifshitz type [26] and its future potential applications in quantum field theory.

## Appendix A: Minimal set of massive integrals in dimensional regularization

Here we compute the minimal set of integrals to be used in Sections III and IV. In the *BPHZ* method, the results which are going to be demonstrated should be complemented with additional information which is the content of Appendix B.

Since this computation is well known from textbooks, we shall try to reduce the number of steps in getting the solution of the integrals in an attempt to fix our conventions in a more or less self-contained form. In the new method, we just need to replace the tree-level bare mass  $\mu_0$  by the three-loop bare mass  $\mu$  in all integrals as explained in the main text. We

shall switch to  $m$  in the *BPHZ* technique discussed in Appendix B and in the appropriate places in the body of the paper.

The integrals connected to the four-point function vertex part diagrams are the one-loop integral  $I_2(k_i)$  from Eq. (15), the trivial two-loop contribution is denoted by  $I_2^2(k_i)$  (Eq. (16)) whereas the nontrivial two-loop correction is named  $I_4(k_i)$  (Eq. (17)), i.e., the first, second and third graphs of  $\Gamma^{(4)}(k_i)$  from Eq. (20).

The one-loop integral is given by

$$I_2(P) = \int d^d q \frac{1}{(q^2 + \mu^2)[(q + P)^2 + \mu^2]}. \quad (\text{A1})$$

We introduce a Feynman parameter  $x$  and use the following useful identity in order to set the notation from Ref. [9] in the computation of all diagrams

$$\int \frac{d^d q}{(q^2 + 2k \cdot q + m^2)^\alpha} = \frac{1}{2} \frac{\Gamma(\frac{d}{2})\Gamma(\alpha - \frac{d}{2})(m^2 - k^2)^{\frac{d}{2} - \alpha}}{\Gamma(\alpha)} S_d, \quad (\text{A2})$$

where  $S_d$  is the area of the  $d$ -dimensional unit sphere. After expanding  $d = 4 - \epsilon$  in the argument of the  $\Gamma$  function and using the property  $\Gamma(1 + z) = z\Gamma(z)$ , this integral can be rewritten as

$$I_2(P) = \frac{S_d}{\epsilon} \left(1 - \frac{1}{2}\epsilon\right) \int_0^1 dx [x(1-x)P^2 + \mu^2]^{-\epsilon/2}. \quad (\text{A3})$$

Note that everytime we perform a loop integral, the angular factor  $S_d$  is included in the final answer of the integral. Our primitive vertex parts have a rather interesting property: the expansion in the number of loops actually coincides with an expansion in powers of the coupling constant, provided that we factor out the tree-level coupling constant of the four-point vertex part. Thus we can absorb the angular factor in the redefinition of the coupling constant. If we proceed in this way, this is equivalent to divide each loop integral performed by  $S_d$  and this overall factor disappears in the final answer. We shall take this step into account hereafter in all loop integrals. Consequently, last integral becomes

$$I_2(P) = \frac{\mu^{-\epsilon}}{\epsilon} \left[1 - \frac{1}{2}\epsilon - \frac{1}{2}\epsilon \tilde{L}(P)\right], \quad (\text{A4})$$

where

$$\tilde{L}(P) = \int_0^1 dx \ln \left[ x(1-x) \frac{P^2}{\mu^2} + 1 \right]. \quad (\text{A5})$$

When integrals like this (see also  $\tilde{L}_3$  below) are multiplied by inverse powers of  $\epsilon$  they must cancel in the renormalization algorithm.

Since the diagrammatic identity is valid

$$\text{Diagram with two circles connected by two lines} = \left( \text{Diagram with two circles connected by one line} \right)^2, \quad (\text{A6})$$

the integral corresponding to this diagram yields  $I_2^2(P)$  and to the order required can be written in the form

$$I_2^2(P) = \frac{\mu^{-2\epsilon}}{\epsilon^2} [1 - \epsilon - \epsilon \tilde{L}(P)]. \quad (\text{A7})$$

In these integrals we recall that  $P$  corresponds to the three possible combinations  $k_1 + k_2$ ,  $k_1 + k_3$  and  $k_2 + k_3$ .

Finally, one of the graphs pertaining to the nontrivial 2-loop contribution of the four-point function is given by

$$I_4(k_i) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + \mu^2)[(P - q_1)^2 + \mu^2](q_2^2 + \mu^2)[(q_1 - q_2 + k_3)^2 + \mu^2]}, \quad (\text{A8})$$

where  $P = k_1 + k_2$ . Although there are five more diagrams of this type contributing, we stick to this particular distribution of external momenta. After introducing a Feynman parameter and integrating over  $q_2$  we find

$$I_4(k_i) = \frac{1}{\epsilon} \left( 1 - \frac{1}{2}\epsilon \right) \int_0^1 dx [x(1-x)]^{-\epsilon/2} \times \int d^d q_1 \frac{1}{(q_1^2 + \mu^2)[(P - q_1)^2 + \mu^2][(q_1 + P_3)^2 + m_x^2]^{\epsilon/2}}, \quad (\text{A9})$$

where

$$m_x^2 = \frac{\mu^2}{x(1-x)}. \quad (\text{A10})$$

Using in sequence two Feynman parameters  $z$  and  $y$  and integrating over  $q_1$ , the integral takes the purely parametric form

$$I_4(k_i) = \frac{1}{4\epsilon} (1 - \epsilon) \int_0^1 dx [x(1-x)]^{-\epsilon/2} \int_0^1 dy (1-y)^{\epsilon/2-1} y \times \int_0^1 dz [yz(1-yz)P^2 + y(1-y)P_3^2 + 2yz(1-y)P_3P + \mu^2 y + m_x^2(1-y)]^{-\epsilon}. \quad (\text{A11})$$

Notice that the parametric integral is divergent at  $y = 1$  when  $\epsilon = 0$ . In this diagram the leading divergences translate themselves as poles of the form  $\frac{1}{\epsilon^2}$  and  $\frac{1}{\epsilon}$ . The term between

brackets which multiply the  $y$  integral can be written as  $\{\}^{-\epsilon} = \{\}^{-\epsilon}|_{y=1} + [\{\}^{-\epsilon} - \{\}^{-\epsilon}|_{y=1}]$ . Next, since  $a^{-\epsilon} = 1 - \epsilon \ln a + O(\epsilon^2)$ , we can write

$$\{\}^{-\epsilon} = \{\}^{-\epsilon}|_{y=1} - \epsilon \ln \left[ \frac{\{\}}{\{\}|_{y=1}} \right]. \quad (\text{A12})$$

Since the logarithm term vanishes when  $y \rightarrow 1$ , the remaining integral multiplied by this term is therefore convergent when  $\epsilon \rightarrow 0$ . The factor of  $\epsilon$  multiplying the logarithm cancels the overall  $\frac{1}{\epsilon}$  coefficient in  $I_4(k_i)$ , contributes  $O(\epsilon^0)$  to that integral and shall be neglected henceforth. Utilizing this procedure, the three parametric integrals can be performed separately and expanding the results in  $\epsilon$  leads to

$$I_4(k_i) = \frac{\mu^{-2\epsilon}}{2\epsilon^2} \left[ 1 - \frac{1}{2}\epsilon - \epsilon \tilde{L}(P) \right]. \quad (\text{A13})$$

Consider now the minimal number of diagrams belonging to the two-point vertex function, namely the integrals appearing in Eqs. (8) and (9). Denote the integral corresponding to Eq. (8) (“sunset”) by  $I_3(P)$ , i.e., the two-loop expression

$$I_3(P) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)[(q_1 + q_2 + P)^2 + \mu^2]}. \quad (\text{A14})$$

Utilizing the partial  $p$  technique defined by the operation (summation convention is implied and  $i = 1, \dots, d$ , since the metric is Euclidean)

$$1 = \frac{1}{2d} \left( \frac{\partial q_1^i}{\partial q_1^i} + \frac{\partial q_2^i}{\partial q_2^i} \right) \quad (\text{A15})$$

we can rewrite last expression as

$$I_3(P) = \frac{1}{2d} \int d^d q_1 d^d q_2 \left( \frac{\partial q_1^\mu}{\partial q_1^\mu} + \frac{\partial q_2^\mu}{\partial q_2^\mu} \right) \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)[(q_1 + q_2 + P)^2 + \mu^2]}. \quad (\text{A16})$$

After integrations by parts and discarding surface terms we are led to

$$I_3(P) = -\frac{1}{d-3} [3\mu^2 A(P) + B(P)], \quad (\text{A17})$$

where

$$A(P) = \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)[(q_1 + q_2 + P)^2 + \mu^2]^2}, \quad (\text{A18a})$$

$$B(P) = \int d^d q_1 d^d q_2 \frac{P \cdot (q_1 + q_2 + P)}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)[(q_1 + q_2 + P)^2 + \mu^2]^2}. \quad (\text{A18b})$$

Let us first work out  $A(P)$ . We redefine the momenta in the following way: first we define a new momentum  $-q_1' = q_1 + q_2$ , such that  $q_1 = -(q_1' + q_2)$ . Taking into account the

invariance of the integral by the exchange  $P \rightarrow -P$ , after redefining back  $q_1' \rightarrow q_1$ ,  $A(P)$  can be expressed in the form

$$A(P) = \int d^d q_1 d^d q_2 \frac{1}{[(q_1 + P)^2 + \mu^2]^2 (q_2^2 + \mu^2) [(q_1 + q_2)^2 + \mu^2]}. \quad (\text{A19})$$

Introducing a Feynman parameter in order to solve the integral over  $q_2$ , using Eq. (A2) and expanding everything in  $d = 4 - \epsilon$  using the identity  $\Gamma(a + b\epsilon) = \Gamma(a)[1 + b\epsilon\psi(a) + \frac{(b\epsilon)^2}{2}(\psi'(a) + \psi^2(a)) + O(\epsilon^3)]$  (with  $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$ ), the last expression becomes

$$A(P) = \frac{1}{\epsilon} \left( 1 - \frac{1}{2}\epsilon + \frac{\pi^2}{24}\epsilon^2 \right) \int_0^1 dx [x(1-x)]^{-\epsilon/2} \int \frac{d^d q_1}{[(q_1 + P)^2 + \mu^2]^2 [q_1^2 + m_x^2]^{\epsilon/2}}. \quad (\text{A20})$$

Employing another Feynman parameter, integrating over  $q_1$  and expanding in  $\epsilon$  as before, we have

$$A(P) = \frac{1}{4\epsilon} \left( 1 - \frac{1}{2}\epsilon + \frac{\pi^2}{24}\epsilon^2 \right) \left( 1 - \frac{1}{2}\epsilon + \frac{\pi^2}{12}\epsilon^2 \right) \int_0^1 dx [x(1-x)]^{-\epsilon/2} \int_0^1 dy y^{\frac{\epsilon}{2}-1} \times \\ (1-y) \left\{ y(1-y)P^2 + \left[ 1 - y + \frac{y}{x(1-x)} \right] \mu^2 \right\}^{-\epsilon}. \quad (\text{A21})$$

We wish to compute this integral up to its regular terms. Now, using the identity  $y^{\frac{\epsilon}{2}-1} = \frac{2}{\epsilon} \frac{dy}{dy} y^{\frac{\epsilon}{2}}$ , integrating by parts over  $y$  and keeping up to  $O(\epsilon^2)$  terms, we find

$$A(P) = \frac{\mu^{-2\epsilon}}{2\epsilon^2} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) \epsilon^2 \right) + \frac{\mu^{-2\epsilon}}{4} \tilde{i}(P), \quad (\text{A22})$$

where

$$\tilde{i}(P) = \int_0^1 dx \int_0^1 dy \ln y \frac{d}{dy} \left( (1-y) \ln \left[ y(1-y) \frac{P^2}{\mu^2} + 1 - y + \frac{y}{x(1-x)} \right] \right). \quad (\text{A23})$$

The integral  $B(P)$  can be rewritten as

$$B(P) = -\frac{1}{2} P^i \frac{\partial}{\partial P^i} \int d^d q_1 d^d q_2 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)[(q_1 + q_2 + P)^2 + \mu^2]}. \quad (\text{A24})$$

Integrating over  $q_2$  and performing a change of variables, we obtain

$$B(P) = -\frac{1}{2\epsilon} \left( 1 - \frac{1}{2}\epsilon \right) P^i \frac{\partial}{\partial P^i} \int_0^1 dx [x(1-x)]^{-\epsilon/2} \times \\ \int d^d q_1 \frac{1}{[(q_1 + P)^2 + \mu^2](q_1^2 + m_x^2)^{\epsilon/2}}. \quad (\text{A25})$$

Integration over  $q_1$  followed by the expansion  $d = 4 - \epsilon$  yields the result

$$B(P) = \frac{P^2}{4\epsilon} (1 - \epsilon) \int_0^1 dx [x(1-x)]^{-\epsilon/2} \int_0^1 dy y^{\epsilon/2} (1-y) \times \\ \left\{ y(1-y)P^2 + \left[ 1 - y + \frac{y}{x(1-x)} \right] \mu^2 \right\}^{-\epsilon}. \quad (\text{A26})$$

We expand the last bracket for small  $\epsilon$  and when the parametric integrals are carried out, we find

$$B(P) = \frac{P^2 \mu^{-2\epsilon}}{8\epsilon} \left[ 1 - \frac{3}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right]. \quad (\text{A27})$$

where

$$\tilde{L}_3(P, \mu) = \int_0^1 dx dy (1-y) \ln \left\{ y(1-y) \frac{P^2}{\mu^2} + 1 - y + \frac{y}{x(1-x)} \right\}. \quad (\text{A28})$$

Consequently, the integral corresponding to the sunset diagram can be written as

$$I_3(P) = \mu^{-2\epsilon} \left\{ -\frac{3\mu^2}{2\epsilon^2} \left[ 1 + \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + 1 \right) \right] - \frac{3\mu^2}{4} \tilde{i}(P) - \frac{P^2}{8\epsilon} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] \right\}. \quad (\text{A29})$$

This result implies that

$$I_3(P) - I_3(P=0) = \mu^{-2\epsilon} \left\{ -\frac{P^2}{8\epsilon} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] - \frac{3\mu^2}{4} \tilde{I}(P) \right\}, \quad (\text{A30})$$

where

$$\tilde{I}(P) = \int_0^1 dx \int_0^1 dy \ln y \frac{d}{dy} \left( (1-y) \ln \left[ \frac{y(1-y) \frac{P^2}{\mu^2} + 1 - y + \frac{y}{x(1-x)}}{1 - y + \frac{y}{x(1-x)}} \right] \right). \quad (\text{A31})$$

This turns out to furnish the following diagrammatic expression:

$$\left( \text{Diagram 1} \Big|_{\mu} - \text{Diagram 2} \Big|_{k=0, \mu} \right) = \frac{N+2}{3} \mu^{-2\epsilon} \left\{ -\frac{P^2}{8\epsilon} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] - \frac{3\mu^2}{4} \tilde{I}(P) \right\}. \quad (\text{A32})$$

The integral  $I_5(P)$  is defined by (see Eq. (9))

$$I_5(P) = \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \frac{1}{[(q_1 + q_2 + P)^2 + \mu^2][(q_1 + q_3 + P)^2 + \mu^2]}. \quad (\text{A33})$$

The appropriate version of the partial  $p$  procedure is now

$$1 = \frac{1}{3d} \left( \frac{\partial q_1^i}{\partial q_1^i} + \frac{\partial q_2^i}{\partial q_2^i} + \frac{\partial q_3^i}{\partial q_3^i} \right). \quad (\text{A34})$$



Insert this identity inside the integrand, integrate by parts, get rid of surface terms and after some rearrangements we can write

$$I_5(P) = -\frac{2}{3d-10}[5\mu^2 C(P) + D(P)], \quad (\text{A35})$$

where

$$C(P) = \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \frac{1}{[(q_1 + q_2 + P)^2 + \mu^2][(q_1 + q_3 + P)^2 + \mu^2]}, \quad (\text{A36a})$$

$$D(P) = \left(-\frac{1}{2}P^i \frac{\partial}{\partial P^i}\right) \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{(q_1^2 + \mu^2)(q_2^2 + \mu^2)(q_3^2 + \mu^2)} \times \frac{1}{[(q_1 + q_2 + P)^2 + \mu^2][(q_1 + q_3 + P)^2 + \mu^2]}. \quad (\text{A36b})$$

Performing the replacement  $q_1 + P = q'_1$ , restoring  $q'_1 \rightarrow q_1$  and  $P \rightarrow -P$  just as we did before in the computation of  $I_3(P)$ , we have

$$C(P) = \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{[(q_1 + P)^2 + \mu^2]^2} \times \frac{1}{(q_2^2 + \mu^2)(q_3^2 + \mu^2)[(q_1 + q_2)^2 + \mu^2][(q_1 + q_3)^2 + \mu^2]}, \quad (\text{A37a})$$

$$D(P) = -\frac{1}{2}P^i \frac{\partial}{\partial P^i} \int d^d q_1 d^d q_2 d^d q_3 \frac{1}{[(q_1 + P)^2 + \mu^2]} \times \frac{1}{(q_2^2 + \mu^2)(q_3^2 + \mu^2)[(q_1 + q_2)^2 + \mu^2][(q_1 + q_3)^2 + \mu^2]}. \quad (\text{A37b})$$

The object  $C(P)$  can be rewritten in the form

$$C(P) = \int d^d q_1 \frac{1}{[(q_1 + P)^2 + \mu^2]^2} \times \left( \int d^d q_2 \frac{1}{(q_2^2 + \mu^2)[(q_1 + q_2)^2 + \mu^2]} \right)^2. \quad (\text{A38})$$

Now, following similar steps as those employed in the computation of  $A(P)$ , we get to the following result:

$$C(P) = \frac{\mu^{-3\epsilon}}{3\epsilon^3} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{24} + \frac{9}{4} \right) \epsilon^2 \right) + \frac{\mu^{-3\epsilon}}{2\epsilon} \tilde{i}(P), \quad (\text{A39})$$

where  $\tilde{i}(P)$  is given by Eq. (A23).

The integral  $D(P)$  can be performed analogously and its singular part within the  $\epsilon$  expansion reads:

$$D(P) = \frac{P^2 \mu^{-3\epsilon}}{6\epsilon^2} \left[ 1 - \epsilon - 3\epsilon \tilde{L}_3(P, \mu) \right]. \quad (\text{A40})$$

We can now express the three-loop integral contributing to the two-point function in the form

$$I_5(P) = \mu^{-3\epsilon} \left\{ -\frac{5\mu^2}{3\epsilon^3} \left[ 1 + \epsilon + \left( \frac{\pi^2}{24} + \frac{15}{4} \right) \epsilon^2 \right] - \frac{5\mu^2}{2\epsilon} \tilde{i}(P) - \frac{P^2}{6\epsilon^2} \left[ 1 + \frac{1}{2}\epsilon - 3\epsilon \tilde{L}_3(P, \mu) \right] \right\}. \quad (\text{A41})$$

A useful quantity to our purposes is the difference between this integral computed at arbitrary external momentum from its value at  $P = 0$ , namely

$$I_5(P) - I_5(P = 0) = \mu^{-3\epsilon} \left\{ -\frac{P^2}{6\epsilon^2} \left[ 1 + \frac{1}{4}\epsilon - 2\epsilon \tilde{L}_3(P, \mu) \right] - \frac{5\mu^2}{2\epsilon} \tilde{I}(P) \right\}, \quad (\text{A42})$$

where the remaining integrals are the same as before, Eqs. (A31). In summary, this results shall be useful in the definition of our unconventional minimal subtraction in section III, but also in the standard BPHZ method using minimal subtraction. The latter requires, however, all diagrams from  $\Gamma^{(2)}$ ,  $\Gamma^{(4)}$ . Further details can be found in the main text in connection with simplified versions of some results derived in this Appendix.

## Appendix B: Computation of integrals useful in the BPHZ method

In this Appendix we shall calculate only three-loop diagrams which are momentum-dependent. The relevant graphs to be determined consist of the “sunset” with a tadpole insertion and its counterterm, which is the sunset with a mass coupling constant generated iteratively from the BPHZ framework. As explicitly discussed in the body of the paper, all other three-loop “tadpole diagrams” along with their counterterms do not depend on the external momenta. We shall not be concerned with their explicit calculation since they do not contribute to the field renormalization constant at three-loop order.

Our aim will be modest here. We shall simply show that the singular part of the interesting diagram combined with its counterterm is momentum-independent. First, notice that the combination required of the two-point function of the diagram along with its tadpole is the following

$$u^3 \left[ \frac{\mu^{3\epsilon}}{4} \text{diagram}_1 + \frac{\mu^{2\epsilon} m^2 \tilde{\lambda}_{m^2}}{4u} \text{diagram}_2 \right]. \quad (\text{B1})$$

The first diagram, the “sunset with an inserted tadpole”, is given by the following expression

$$\text{diagram}_1 = \left[ \frac{(N+2)}{3} \right]^2 i_{st}, \quad (\text{B2})$$

whose associated integral reads

$$i_{st} = \int \frac{d^d q_1 d^d q_2 d^d q_3}{(q_1^2 + m^2)^2 (q_2^2 + m^2) ((q_1 + q_2 + k)^2 + m^2) (q_3^2 + m^2)}. \quad (\text{B3})$$

After integrating over  $q_3$ , it can be rewritten as

$$i_{st} = -\frac{m^{2-\epsilon}}{\epsilon} \int \frac{d^d q_1 d^d q_2}{(q_1^2 + m^2)^2 (q_2^2 + m^2) ((q_1 + q_2 + k)^2 + m^2)}. \quad (\text{B4})$$

Note that the remaining integral can be identified (after some reshuffling of the momenta) with  $A(k)$  given by Eq. (A19) when we replace  $\mu \rightarrow m$ . According to Eq. (A22), we then obtain the following intermediate step for the diagram

$$i_{st} = -\frac{m^{2-3\epsilon}}{2\epsilon^3} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) \epsilon^2 \right) - \frac{m^{2-3\epsilon}}{4\epsilon} \tilde{i}(k). \quad (\text{B5})$$

Therefore, the total contribution of the first diagram is

$$\begin{aligned} \frac{\mu^{3\epsilon}}{4} \text{Diagram} &= -m^2 \left( \frac{m}{\mu} \right)^{-3\epsilon} \frac{(N+2)^2}{72} \left[ \frac{1}{\epsilon^3} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) \epsilon^2 \right) \right. \\ &\quad \left. + \frac{1}{2\epsilon} \tilde{i}(k) \right]. \end{aligned} \quad (\text{B6})$$

The counterterm diagram turns out to be written as

$$\text{Diagram} = \frac{(N+2)}{3} A(k). \quad (\text{B7})$$

After replacing the value  $\tilde{\lambda}_{m^2} = u \delta_{m^2}^{(1)} = \frac{(N+2)u}{6\epsilon}$  and using the previous expression for  $A(k)$  Eq. (A22), the overall contribution of the counterterm is easy to determine, namely

$$\begin{aligned} \frac{\mu^{2\epsilon} m^2 \tilde{\lambda}_{m^2}}{4u} \text{Diagram} &= m^2 \left( \frac{m}{\mu} \right)^{-2\epsilon} \frac{(N+2)^2}{72} \left[ \frac{1}{\epsilon^3} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) \epsilon^2 \right) \right. \\ &\quad \left. + \frac{1}{2\epsilon} \tilde{i}(k) \right]. \end{aligned} \quad (\text{B8})$$

Therefore, summing up (B6) and (B8), the singular terms which depend on the external momenta exactly cancel each other. Indeed, performing explicitly the summation, the aforementioned combination of these two diagrams yields

$$\begin{aligned} u^3 \left[ \frac{\mu^{3\epsilon}}{4} \text{Diagram} + \frac{\mu^{2\epsilon} m^2 \tilde{\lambda}_{m^2}}{4u} \text{Diagram} \right] &= m^2 \ln \left( \frac{m}{\mu} \right) \frac{(N+2)^2}{72} \left[ \frac{1}{\epsilon^2} \left( 1 - \frac{1}{2}\epsilon + \left( \frac{\pi^2}{12} + \frac{1}{2} \right) \epsilon^2 \right) \right. \\ &\quad \left. \times \left\{ 1 - \frac{5\epsilon}{2} \ln \left( \frac{m}{\mu} \right) \right\} \right]. \end{aligned} \quad (\text{B9})$$

Although there are singular terms proportional to  $\ln\left[\frac{m}{\mu}\right]$  as explained before, the contributions coming from all three-loop diagrams of the two-point vertex part shall eliminate them, precisely as we have shown explicitly at two-loop level, although we do not pursue this proof herein. Since our goal is just to collect singular terms which are explicitly momentum-dependent for the reasons explained in the main text, the terms multiplying  $\ln\left[\frac{m}{\mu}\right]$  can be safely neglected in the computation of the field normalization function at three-loop order. This concludes our task.

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